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A COMPARISON OF DIFFUSION THEORY AND TRANSPORT THEORY RESULTS FOR THE PENETRATION OF RADIATION INTO PLANE SEMI-INFINITE SLABS

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ABSTRACT

The penetration of radiation into plane semi-infinite slabs of material has been calculated numerically by means of the time-dependent transport theory (Boltzmann equation) and the approximate "diffusion theory." Quantitative results are given for the case of a non-scattering material with constant absorption cross section, and for the general case of a scattering material (boron) with absorption coefficient a function of energy. Results show that the diffusion theory calculation gives an energy penetration which is too large, but that in a time during which the diffusion wave penetrates to a depth corresponding to a few mean free paths, the <u>rate</u> of energy penetration calculated using the diffusion approximation approaches the value calculated using the exact theory.

ACKNOWLEDGMENT

The following personnel of Group T-1 assisted with programming the transport theory calculations for the IBM 701 computer and with details of operating the machine: Donald Monk, S. I. Schlesinger, Eugene Willbanks. Eugene Herbst of Group T-7 programmed the diffusion equation calculation for the MANIAC.

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The diffusion of photons through matter is described by the time-dependent Boltzmann transport equation:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\vec{\omega}\cdot\nabla)I_\nu = \sigma_a(1-e^{-\varepsilon})\left[J_\nu(T) - I_\nu\right]$$

$$- \sigma_s I_\nu + \int \sigma_s(\vec{\omega},\vec{\omega}')I_\nu(\vec{\omega}')d\omega' \qquad (1)$$

In this equation $\varepsilon = h\nu/T$; $\vec{\omega}$ is a unit vector in the direction of motion of the beam of photons of energy $h\nu$ passing through matter at temperature T. The quantity I_ν is the monochromatic radiation intensity, i.e., $I_\nu(\vec{R},t,\nu,\vec{\omega})d\omega d\nu/h\nu$ is the density of photons at $(\vec{R},t)$ moving in directions within $d\omega$ about $\vec{\omega}$ with energies between $h\nu$ and $h(\nu+d\nu)$. The energy density of the radiation field at $(\vec{R},t)$ is

$$E_r = \int d\omega \int_0^\infty I_\nu(\nu,\vec{\omega})d\nu .$$

The first term on the right-hand side of eq (1) represents the contribution to the beam by emission minus absorption. $\sigma_a(\nu,T)$ = cross section/cm^3 for absorption of photons by the matter at $(\vec{R},t)$. In the case of local thermodynamic equilibrium assumed here, the emission is described by the Planck function,

$$J_\nu(T) = \frac{2h}{c^3}\,\frac{\nu^3}{e^\varepsilon - 1} .$$

The presence of the $e^{-\varepsilon}$ factor in eq (1) is due to the effect of induced emission. The last two terms in the right-hand member of (1) describe the change in the beam due to scattering of photons by the

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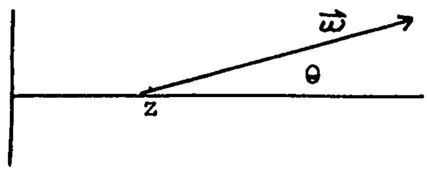
matter at $(\vec{R},t)$.

For the complete determination of the spatial and time variation of $I_\nu(\vec{R},t,\nu,\vec{\omega})$ and $T(\vec{R},t)$, one other equation (together with appropriate boundary conditions) is needed. The required equation expresses the overall conservation of energy in the system matter plus radiation, and may be written

$$\frac{\partial E_m(T)}{\partial t} + \frac{\partial E_r}{\partial t} + \nabla\cdot\vec{F} = 0 \tag{2}$$

where $\vec{F}$ is the net flux of radiation. The latter quantity is a vector whose component in a direction $\vec{\ell}$ is given by $c\int d\omega\int_0^\infty I_\nu(\vec{\omega})\cos(\vec{\omega},\vec{\ell})d\nu$.



Diffusion Approximation.

Specialized to the case of a semi-infinite plane slab (with axial symmetry), the transport equation (1) is written,

$$\frac{1}{c}\frac{\partial}{\partial t} I_\nu(z,t,\nu,\mu) + \mu\frac{\partial I_\nu}{\partial z} =$$

$$\sigma_a(1-e^{-\varepsilon})\left[J_\nu(T)-I_\nu\right] - \sigma_s I_\nu + \int\sigma_s(\vec{\omega},\vec{\omega}')I_\nu(\vec{\omega}')d\omega'. \tag{3}$$

Here $\mu = \cos\theta$ (see figure). In the diffusion approximation the distribution function is considered to be nearly isotropic:

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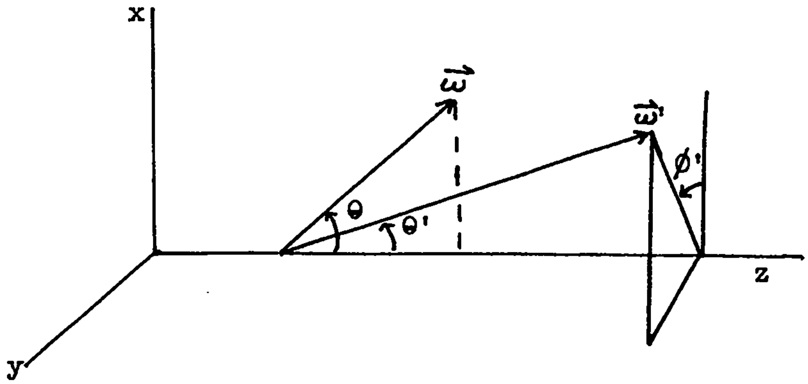
$$I_\nu(z,t,\mu) = a_o(z,t) + a_1(z,t)\mu \, . \tag{4}$$

In the case of Thomson scattering by electrons at rest, which is applicable here, the scattering kernel is given by

$$\sigma_s(\vec{\omega},\vec{\omega}') = \frac{1}{2} N r_o^2 \left[1 + \cos^2(\vec{\omega},\vec{\omega}') \right].$$

The scattering integral is to be evaluated when I_ν is of the form given by (4) above. Taking coordinates as in the figure, the direction of



$\vec{\omega}'$ can be represented by ($\sin\theta' \cos\phi'$, $\sin\theta' \sin\phi'$, $\cos\theta'$). Because of the symmetry, $\vec{\omega}$ can be taken in the (x,z)-plane: $\vec{\omega}$ = ($\sin\theta$, 0, $\cos\theta$); also $I_\nu(\vec{\omega}') = I_\nu(\theta')$. The scattering integral is then given by

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$$\frac{1}{2} Nr_o^2 \int I_\nu (\theta') \left[1 + \cos^2(\vec{\omega}, \vec{\omega}')\right] d\omega'$$

$$= \frac{Nr_o^2}{2} \int_o^{2\pi} \int_o^{\pi} I_\nu(\theta') \left[1 + (\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\varphi')^2\right] \sin\theta' \, d\theta' \, d\varphi'$$

$$= \frac{\pi Nr_o^2}{2} \left\{ (3-\mu^2) \int_{-1}^{1} I_\nu(\mu') d\mu' + (3\mu^2 - 1) \int_{-1}^{1} (\mu')^2 I_\nu(\mu') d\mu' \right\}$$

where $\mu = \cos\theta$. Evaluation of this expression with $I_\nu(\mu')$ given by (4) gives $\frac{8}{3} \pi Nr_o^2 a_o$ for the scattering integral. Since $\sigma_s = \frac{8}{3} \pi Nr_o^2$ (total cross-section), we have

$$\int_\omega \sigma_s(\vec{\omega}, \vec{\omega}') I_\nu (\vec{\omega}') d\omega' = \sigma_s \cdot a_o \qquad (5)$$

in this approximation.

Substitution of the expression (4) for I_ν in the transport equation (3), taking account of (5) and neglecting the time derivative in (3), gives the following relation:

$$\mu \frac{\partial a_o}{\partial z} + \mu^2 \frac{\partial a_1}{\partial z} = \sigma_a (1 - e^{-\varepsilon}) \left[J_\nu(T) - (a_o + a_1 \mu) \right]$$

$$- \sigma_s (a_o + a_1 \mu) + \sigma_s a_o .$$

Since this relation must hold for all values of μ, the coefficients of corresponding powers of μ can be equated. This gives

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$$\left.\begin{aligned} a_o &= J_\nu(T) \\ \frac{\partial a_o}{\partial z} &= -\left[\sigma_a(1-e^{-\varepsilon}) + \sigma_s\right] a_1 \end{aligned}\right\}$$

Thus, the form of I_ν in the diffusion theory is

$$I_\nu(\mu) = J_\nu(T) - \lambda_\nu \mu \frac{\partial J_\nu}{\partial z} \tag{6}$$

where

$$\frac{1}{\lambda_\nu} = \sigma_a(1-e^{-\varepsilon}) + \sigma_s.$$

The radiation energy density becomes

$$E_r = 2\pi \int_o^\infty \int_{-1}^1 I_\nu \, d\mu \, d\nu = 4\pi \int_o^\infty J_\nu \, d\nu = aT^4 \tag{7}$$

where $\underline{a}$ = Stefan-Boltzmann constant.

The net flux of radiation at (z,t) is a vector in the direction of the z-axis with magnitude

$$F = 2\pi c \int_{-1}^1 \int_o^\infty \mu I_\nu(\mu) d\nu \, d\mu.$$

Using (6), the diffusion approximation expression for the flux is obtained:

$$F = -\frac{4\pi}{3} c \int_o^\infty \lambda_\nu \frac{\partial J_\nu(T)}{\partial T} d\nu \cdot \frac{\partial T}{\partial z} \quad .$$

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Introducing the "Rosseland mean" defined by

$$\bar{\lambda}(T) = \frac{\int_0^\infty \lambda_\nu \frac{\partial J_\nu}{\partial T}\, d\nu}{\int_0^\infty \frac{\partial J_\nu}{\partial T}\, d\nu} = \frac{\pi}{aT^3} \int_0^\infty \lambda_\nu \frac{\partial J_\nu}{\partial T}\, d\nu \,,$$

the flux becomes simply

$$F = -\frac{c\bar{\lambda}}{3} \frac{\partial}{\partial z} (aT^4) \tag{8}$$

in the diffusion approximation.

When the approximate relations (7), (8) for E_r and F are used in the energy balance equation (2), the time-dependent diffusion equation is obtained:

$$\frac{\partial}{\partial t}\left[E_m(T) + aT^4\right] = \frac{\partial}{\partial z}\left[\frac{c\bar{\lambda}}{3} \frac{\partial}{\partial z}(aT^4)\right] . \tag{9}$$

The above derivation indicates the nature of the approximations inherent in the diffusion theory.

<u>Comparison of Exact and Diffusion Theory by Numerical Solutions; Boundary Conditions.</u>

In order to gain some quantitative insight into the difference between the exact theory and diffusion approximation treatments of the radiation transport process, a simple problem involving penetration of radiation into a semi-infinite slab of cold incompressible material was

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devised and the solutions corresponding to the exact and diffusion approximation treatments were obtained (numerically). It is assumed that the surface of the slab is illuminated by isotropic radiation with a Planck frequency distribution corresponding to constant temperature, T_o. The temperature dependence of the energy of the material in the slab ($z \geqslant 0$) is assumed to be of the form

$$E_m(T) = bT$$

where b is a constant. In the first problem the material is assumed to be non-scattering with λ = constant independent of ν,T. The transport equation (3) can then be integrated over $d\nu$ to give

$$\frac{1}{c}\,\frac{\partial\psi}{\partial t} + \mu\,\frac{\partial\psi}{\partial z} = \frac{1}{\lambda}\left(\frac{1}{2}\,aT^4 - \psi\right) \tag{10}$$

where

$$\psi(z,t,\mu) = 2\pi\int_0^\infty I_\nu\,d\nu\,.$$

The radiation energy is given by

$$E_r = \int_{-1}^{1}\psi\,d\mu \tag{11}$$

and the energy relation (2) becomes

$$b\,\frac{\partial T}{\partial t} = -\,\frac{\partial}{\partial t}\int_{-1}^{1}\psi d\mu\; - c\,\frac{\partial}{\partial z}\int_{-1}^{1}\mu\psi d\mu\,.$$

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By integrating (10) over $d\mu$ the last result can be written,

$$b \frac{\partial T}{\partial t} = \frac{c}{\lambda} \left(\int_{-1}^{1} \psi \, d\mu - aT^4 \right) \tag{12}$$

The solution of (10) and (12) is then to be obtained subject to the boundary conditions

$$\left. \begin{aligned} T(z,t) &= 0 \\ \psi(z,t,\mu) &= 0 \end{aligned} \right\} \quad \text{for } z > 0 \text{ when } t = 0, \tag{13}$$

$$\psi(z,t,\mu) = \frac{1}{2} aT_o^{\,4} \quad \text{at } z = 0 \text{ for } 0 < \mu < 1.$$

The solution of the penetration problem in the diffusion theory approximation is obtained by integrating the diffusion equation

$$\frac{\partial}{\partial t} (bT + aT^4) = \frac{c\lambda}{3} \frac{\partial^2}{\partial z^2} (aT^4) \tag{14}$$

subject to a boundary condition prescribed at the surface z = 0 and the initial condition

$$T(z,0) = 0, \qquad (z > 0). \tag{15}$$

In the diffusion theory treatment it is evidently not possible to specify the value of the intensity ψ at the boundary of the slab for $0 < \mu < 1$ as was done in the transport theory formulation. An alternative procedure that is frequently employed in this type of flow problem is to specify T at the boundary:

$$T(0,t) = T_o = \text{const.} \tag{16a}$$

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However, this is only an approximate description; the transport theory solution shows that the temperature of the surface of the slab rises continuously and approaches T_o as the material of the slab heats up.

A description which accords closer agreement with the transport theory formulation is to prescribe the value of the forward current at the boundary:

$$F^+ = c\int_0^1 \mu\psi(\mu)d\mu = \frac{1}{4}caT_o^4 \qquad \text{at } z = 0.$$

By (6) this is equivalent to writing

$$\left[\frac{aT^4}{2} - \frac{\bar{\lambda}}{3}\frac{\partial aT^4}{\partial z}\right]_{z=0} = \frac{aT_o^{\ 4}}{2} \quad . \tag{16b}$$

For the purpose of comparing the diffusion theory treatment with the exact solution, numerical solutions were obtained using both kinds of approximate boundary conditions*.

It will be noted that if z and ct are measured in units of λ, the problem here described involves only one parameter: $\eta = aT_o^{\ 3}/b$. Thus, if $b \dashrightarrow \alpha b$ and $T_o^{\ 3} \dashrightarrow \alpha T_o^{\ 3}$, the corresponding solution of (10), (12) is

$$T \dashrightarrow \alpha^{1/3}T$$
$$\psi \dashrightarrow \alpha^{4/3}\psi$$

*For a discussion of boundary conditions for the steady-state diffusion problem, see Morse and Feshbach, Methods of Theoretical Physics, pp. 185-188 (1953).

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These scaling relations also hold for eq (14).

Numerical Solutions.

The "method of discrete ordinates"* was employed for the numerical solution of the transport problem. In this method the distribution function is calculated at each of a finite set of directions; thus

$$\psi_i = \psi(\mu_i) \qquad i = 1, 2, 3, \ldots, n.$$

The transport equation (10) is replaced by a set of n equations:

$$\frac{1}{c}\frac{\partial \psi_i}{\partial t} + \mu_i \frac{\partial \psi_i}{\partial z} = \frac{1}{\lambda}\left(\frac{1}{2}\, aT^4 - \psi_i\right) \tag{10'}$$

and the integral over directions appearing in (12) is replaced by a numerical quadrature so that (12) becomes

$$\frac{b}{c}\frac{\partial T}{\partial t} = \frac{1}{\lambda}\left(\sum_{i=1}^{n} a_i \psi_i - aT^4\right) \tag{12'}$$

where the coefficients a_i depend upon the quadrature formula selected. For numerical solution described here the interval $-1 \leq \mu \leq 1$ was split into two Gaussian quadrature intervals, $-1 \leq \mu \leq 0$ and $0 \leq \mu \leq 1$, so that the angles are given by the zeroes of

$$P_{2\ell}(2\mu \pm 1) = 0,$$

where $P_{2\ell}$ is the Legendre polynomial of order 2ℓ, and the a_i's are given by one-half the corresponding usual Gaussian coefficients. The advantage

*Cf. Kourganoff, Basic Methods in Transfer Problems, Chapter III (1952).

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of splitting the interval of integration into two Gaussian intervals lies in the fact that this gives more directions in the region near $\mu = 0$ where $\psi(\mu)$ is changing most rapidly*. Thus in the first approximation there are four directions, $\mu = \pm\frac{1}{2} \pm \frac{1}{2}\left(\frac{\sqrt{3}}{3}\right)$.

Using the notation

$$\psi_{i+} = \psi(+\mu_i) \qquad \mu_i \geqslant 0$$

$$\psi_{i-} = \psi(-\mu_i)$$

and employing the indexes n, j, corresponding to the time and space divisions of the mesh respectively, the system of finite difference equations corresponding to eqs. (10'),(12') may be written**,

$$\frac{\psi_{i+}^{n+1,j} - \psi_{i+}^{n,j}}{c\Delta t} + \mu_i \frac{\psi_{i+}^{n,j} - \psi_{i+}^{n,j-1}}{\Delta z} = \frac{1}{\lambda}\left\{\frac{1}{2} a(T^4)^{n,j} - \psi_{i+}^{n,j}\right\}$$

$$\frac{\psi_{i-}^{n+1,j} - \psi_{i-}^{n,j}}{c\Delta t} - \mu_i \frac{\psi_{i-}^{n,j+1} - \psi_{i-}^{n,j}}{\Delta z} = \frac{1}{\lambda}\left\{\frac{1}{2} a(T^4)^{n,j} - \psi_{i-}^{n,j}\right\} \qquad (17)$$

$$\frac{b}{c}\,\frac{T^{n+1,j} - T^{n,j}}{\Delta t} = \frac{1}{\lambda}\left\{\sum_i a_i\left(\psi_{i+}^{n,j} + \psi_{i-}^{n,j}\right) - a(T^4)^{n,j}\right\}$$

*This "double-Gaussian" method was first used by J. B. Sykes (Monthly Notices of Royal Astronomical Society 111, 377 (1951)).

**This method of centering the space differences was suggested by M. Rosenbluth. When centered this way, the equations are integrated in the direction of motion of the photon beam in each case. A stability analysis indicated that the above system is stable for $c\Delta t < \Delta s$, where $\Delta s = |\Delta z/\mu|$, provided $\Delta z << \lambda$.

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This system was solved in the first approximation (4 directions) subject to the boundary conditions (13) on a high-speed digital computer. The value 1/11.5 was used for $\frac{\Delta z}{\lambda}$ and $\frac{c\Delta t}{\lambda}$; several problems were run corresponding to different values of the parameter $\eta = aT_o{}^3/b$.

As a test problem, 250 cycles were run for $\eta = 1.690$ in the second order approximation (8 directions). The results indicate that the energy which penetrates the cold material in a time corresponding to 200 cycles is given within 0.1% by the first-order approximation solution. 100 cycles were run using intervals one-half the value quoted above; the resulting energy penetration did not differ from that calculated using the larger intervals by more than 1%.

The solution of the diffusion equation (14) subject to the boundary conditions (15) and (16a) was obtained by making use of the "similarity" transformation*

$$T(z,t) = T(\xi)$$

where

$$\xi = z(ct)^{-1/2}.$$

If z and ct are measured in units of λ, eq (14) reduces under this transformation to an ordinary differential equation:

$$V\frac{d^2V}{d\xi^2} + \frac{1}{3}\left(\frac{dV}{d\xi}\right)^2 + \frac{3}{2}\,\xi\,(V+1)\,\frac{dV}{d\xi} = 0, \qquad (18)$$

where $V(\xi) = \frac{4aT^3}{b}$. The boundary conditions (16a) become

$$V(0) = V_o = \text{const.}$$
$$V(\infty) = 0 \qquad (19)$$

*Following Marshak, LA-230.

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The radiation wave represented by the solution of (18) has a sharp "front" at some value $\xi = \xi_o$. The solution may be represented as an expansion in powers of $(\xi_o - \xi)$:

$$\begin{cases} V(\xi) = A_1(\xi_o - \xi) + A_2(\xi_o - \xi)^2 + \cdots & [0 \leq \xi \leq \xi_o] \\ V(\xi) = 0 & [\xi_o \leq \xi] \end{cases}$$

The coefficients are determined by substituting this expansion into (18). This gives

$$A_1 = \frac{9}{2}\,\xi_o$$

$$A_2 = \frac{9}{16}\left(\frac{9}{2}\,\xi_o^2 - 1\right)$$

The quantity ξ_o is determined by the first of the boundary conditions (19). The procedure employed for solving (18) was as follows: A first guess for ξ_o is made by using two terms of the expansion for $V(\xi)$. This value of ξ_o and the corresponding slope $V'(\xi_o) = -\frac{9}{2}\xi_o$ are used to start a numerical integration of (18). The integration is continued back to $\xi = 0$ and the value $V(0)$ obtained is compared with V_o. The guessed value for ξ_o is improved and the process is repeated until the desired value $V(0) = V_o$ is closely approximated.

In the case of the boundary condition represented by (16b) it is not possible to obtain a similarity transformation of the kind described above. In this case the solution of the difference system corresponding to (14) over a two-dimensional mesh was carried out numerically.

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General Case: Absorption and Scattering Coefficients Functions of Energy.

The penetration of thermal radiation into boron was calculated by means of the transport theory and the diffusion approximation. This was done for a boundary temperature $T_o = .7$ kv. Values of the absorption and scattering coefficients for boron at .7 kv and density .404 gm/cm^3 which were used are shown in Fig. 12. (That is, σ_a and σ_s were taken to be independent of temperature.) The value .1653 was taken for $aT_o{}^3/b$. The integration of the transport equation (3) was carried out using four directions and ten frequency groups, the latter corresponding to the values $h\nu/.7 = 1, 2, 3, \ldots, 10$. The frequency integrals were approximated by trapezoidal integration. (Trapezoidal evaluation of $\int_0^\infty J_\nu \, d\nu$ at $T = .7$ using the ten frequency values given above is better than 1%). The Rosseland mean $\bar{\lambda}(T)$ for use in the diffusion theory calculation was calculated and fitted; it is shown in Fig. 13. Boundary condition (16b) was employed in the diffusion theory calculation.

Results.

Constant Mean Free Path Case. The total energy penetration to time t is given by $\int_0^t F(0,t)dt$. Plots of this quantity vs ct/λ for several values of the parameter $\eta = aT_o{}^3/b$ are shown in Figs. 1-3, 8. Curves representing the diffusion approximation and exact solutions are shown. The rate of energy penetration, the net flux across the boundary, is shown in Fig. 4 for a typical case. The shapes of the temperature, radiation energy density, and flux waves are shown in Figs. 5-7, and 10. The

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quantities are plotted against the similarity variable $\xi = zt^{-1/2}$: in these coordinates the solution of the diffusion equation subject to boundary condition (16a) is represented by a <u>single curve</u>. The families of curves representing the transport theory solution are shown for comparison. The diffusion theory results shown in Figs. 1-7 were calculated using boundary condition (16a) whereas Figs. 8-10 refer to diffusion theory results obtained using condition (16b). Thus, Fig. 8 is to be compared with Figs. 1 and 4; likewise, Fig. 10 with Fig. 5. Fig. 9 shows the increase of the temperature of the surface of the slab with time. Fig. 11 shows the angular distribution of the radiation intensity at various penetration depths as given by the transport equation calculation in a typical case.

<u>General Case (Variable Mean Free Path)</u>. The results of the calculations of the penetration of radiation into boron are shown in Figs. 14-20, which are similar to those described above for the constant mean free path case. It will be noticed that in Fig. 14 the net flux vs. time curve shows a jump at ct = 120. At this point the calculation was re-zoned in order to avoid exceeding the storage capacity of the computing machine; the space interval size Δz was increased from .5 cm to 1 cm. In the difference scheme employed the net flux at the boundary is given by (employing the notation of eqs (17)) $c\sum_i a_i(\psi_{i+}^{n,o} - \psi_{i-}^{n,1})\mu_i$. It is apparent that except in the case in which $\partial\psi_{i-}/\partial z$ at the boundary is zero, the net flux as calculated by the foregoing expression will show a small jump when the size of the space zones is changed.

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The jump is an effect of the non-infinitesimal value of Δz; the magnitude of the jump is a measure of the error inherent in the calculation, and approaches zero as $\Delta z \dashrightarrow 0$.

SUMMARY

The results of the calculations show that in each case the energy penetration given by the diffusion approximation is too large. According to the diffusion theory, the rate of energy penetration, the net flux at the surface of the slab, is initially infinite in the case of boundary condition (16a) (constant temperature at surface); in the case of boundary condition (16b) (forward current at surface specified) the diffusion calculation gives an initial flux equal to twice that given by the exact theory. In either case, in a time during which the diffusion wave penetrates to a depth corresponding to a few mean free paths, the rate of energy penetration calculated using the diffusion approximation approaches the value given by the transport equation calculation within the limits of accuracy of the calculations. Thus, after the diffusion wave has penetrated a few mean free paths of slab material, the diffusion theory gives the correct energy penetration except for a constant difference. It will be noted, from Fig. 5 in particular, that at advanced times the transport theory temperature wave steepens and, except for a small precursor, approaches the shape of the diffusion wave.

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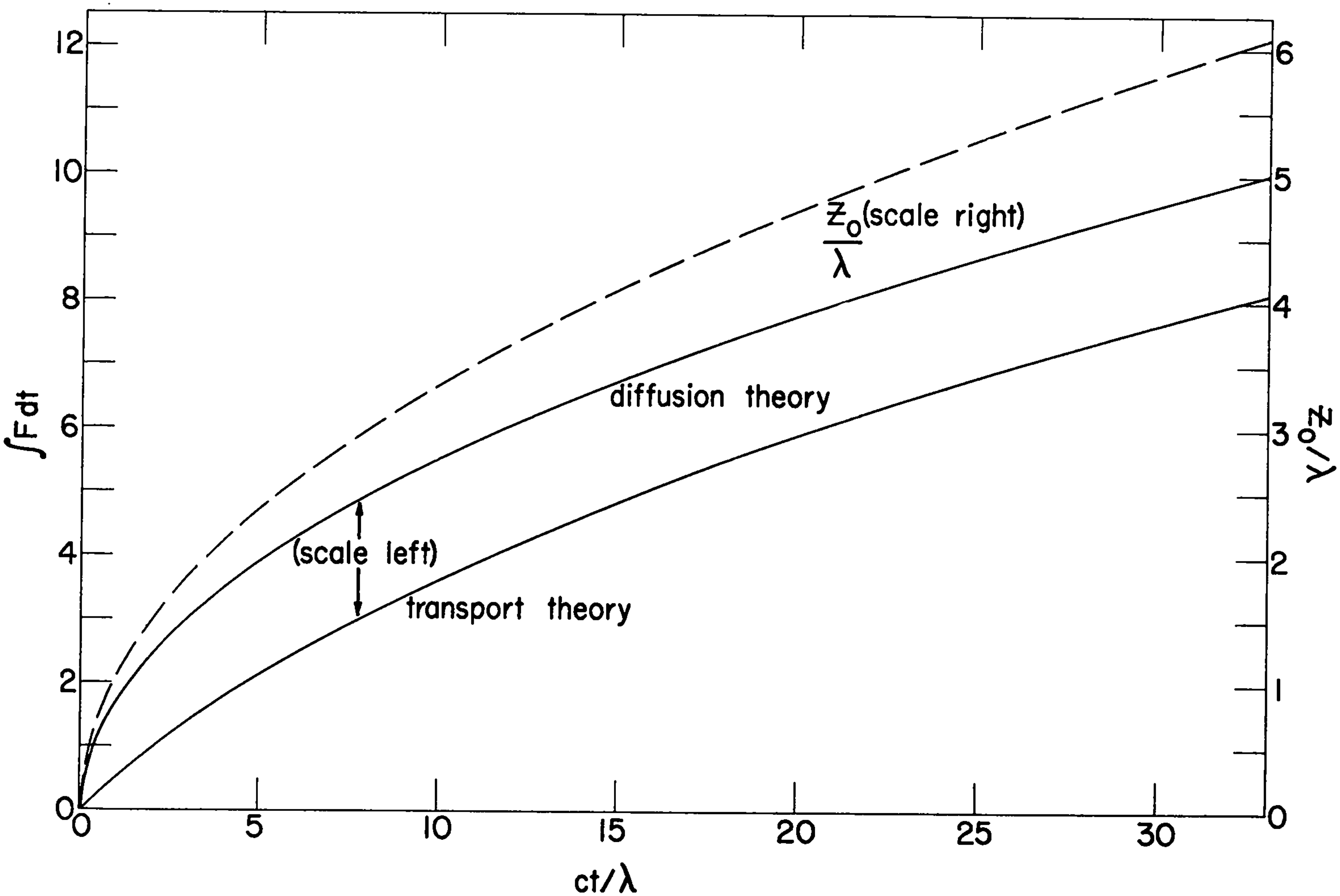

Fig. 1. Energy penetration $\int_0^t F(0,t)dt$ vs ct/λ for constant mean free path case. ($aT_o^3/b = 1.690.$) Unit for $\int Fdt$ is $2.0237\lambda aT_o^4/4$. The boundary temperature was prescribed for the diffusion theory case. The penetration of the diffusion wave is given by $z_o/\lambda = 1.060\ (ct/\lambda)^{1/2}$.

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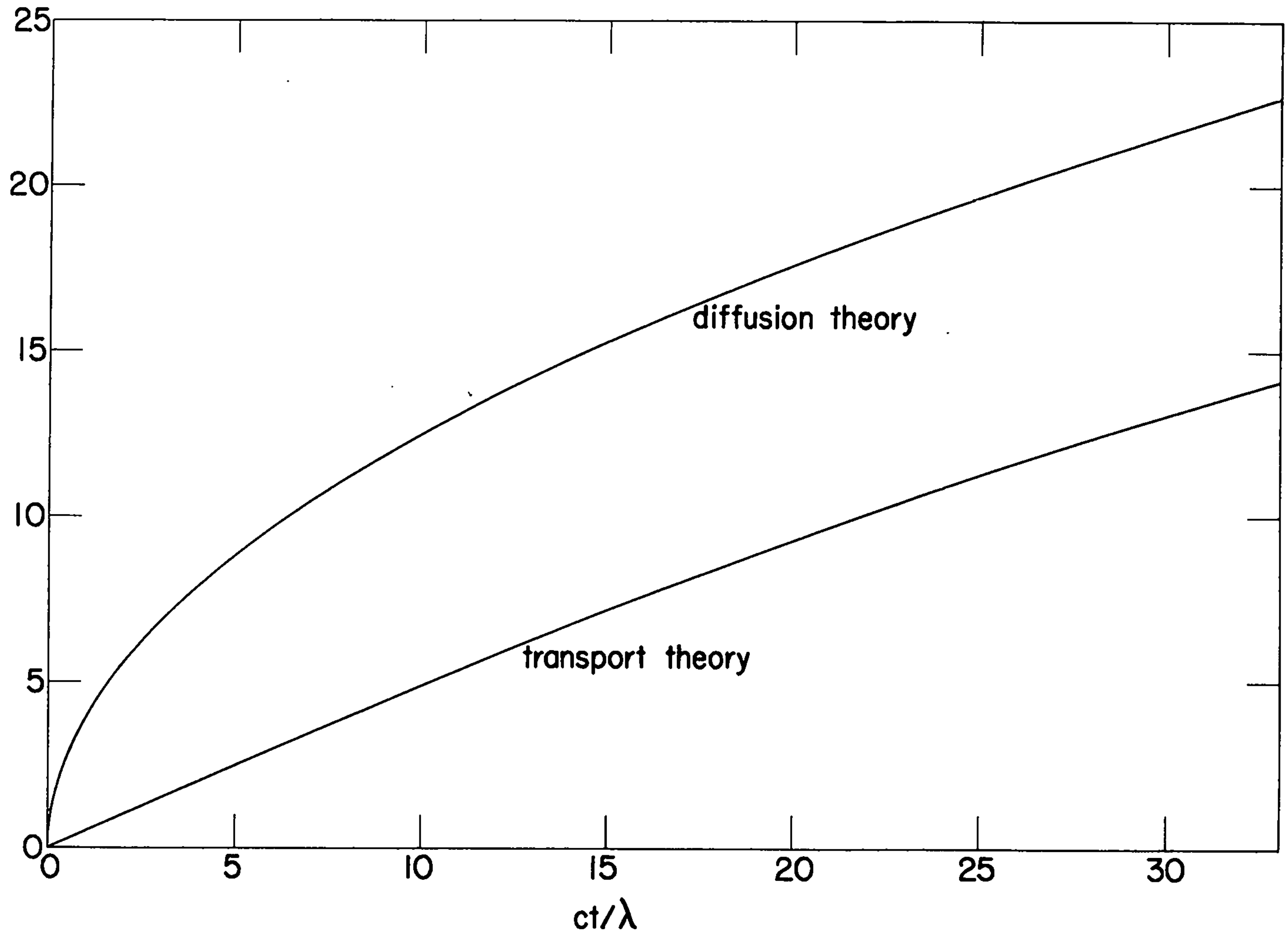

Fig. 2. Energy penetration $\int_0^t F(0,t)dt$ vs ct/λ for constant mean free path case. (aT_o^3/b = .16532.) Unit for $\int Fdt$ is $2.0237\lambda aT_o^4/4$. (Boundary temperature prescribed for diffusion case.) Penetration of the diffusion wave is given by $z_o/\lambda = .3973\,(ct/\lambda)^{1/2}$.

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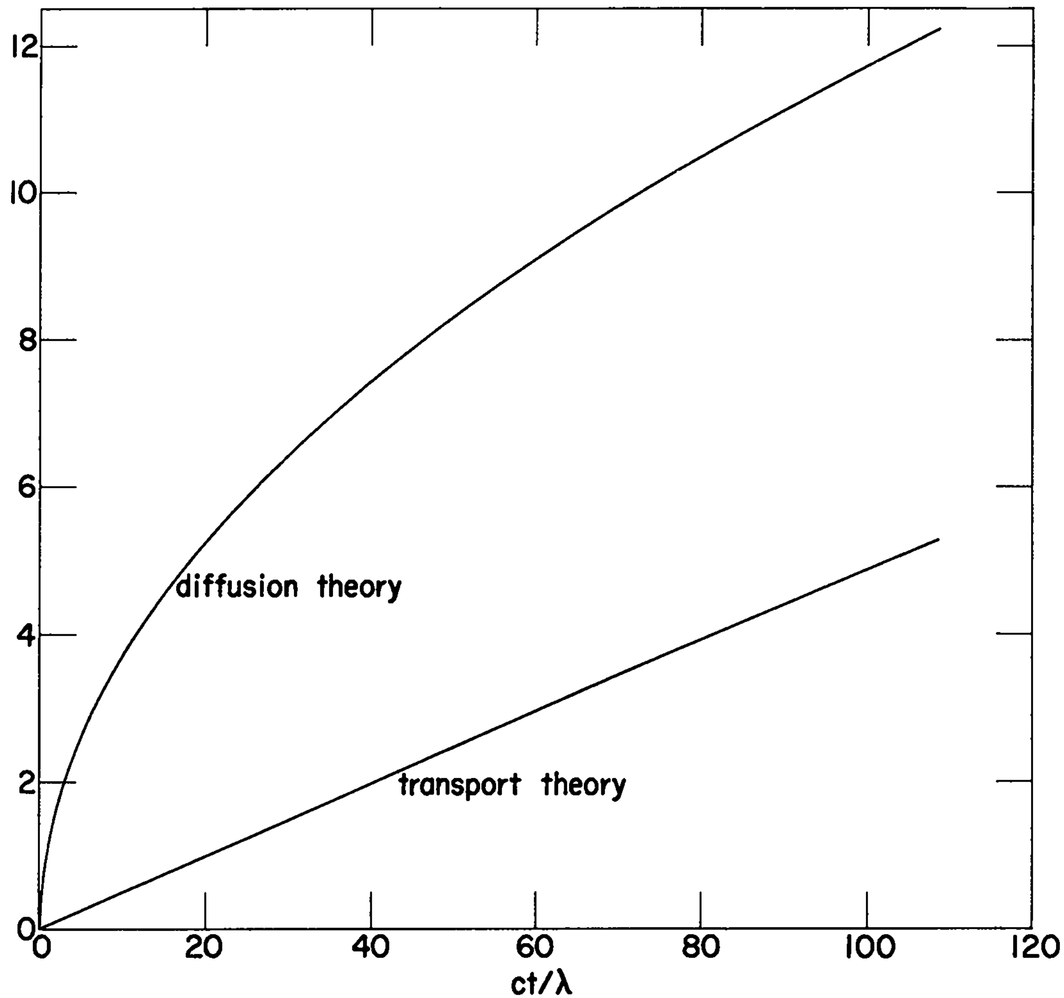

Fig. 3. Energy penetration $\int_0^t F(0,t)dt$ vs ct/λ for constant mean free path case. (aT_o^3/b = .016827). Unit for $\int Fdt$ is $20.311\lambda aT_o^4/4$. (Boundary temperature prescribed for diffusion wave.) The penetration of the diffusion wave is given by $z_o/\lambda = .1300\,(ct/\lambda)^{1/2}$.

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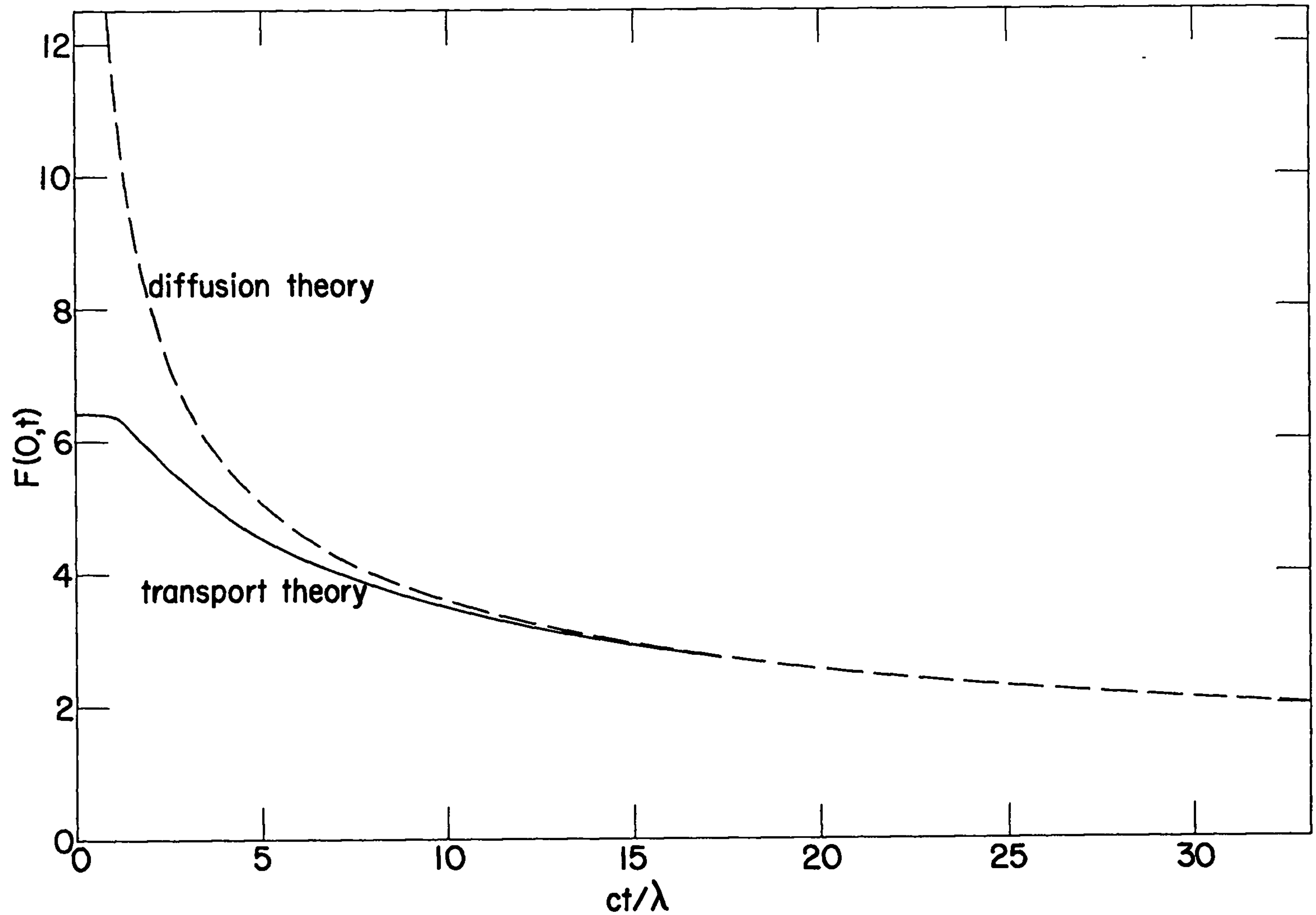

Fig. 4. Net flux through boundary $F(0,t)$ vs ct/λ for constant mean free path case. ($aT_o{}^3/b = 1.690$). Unit for F is $.1557caT_o{}^4/4$. (Boundary temperature prescribed for diffusion case.)

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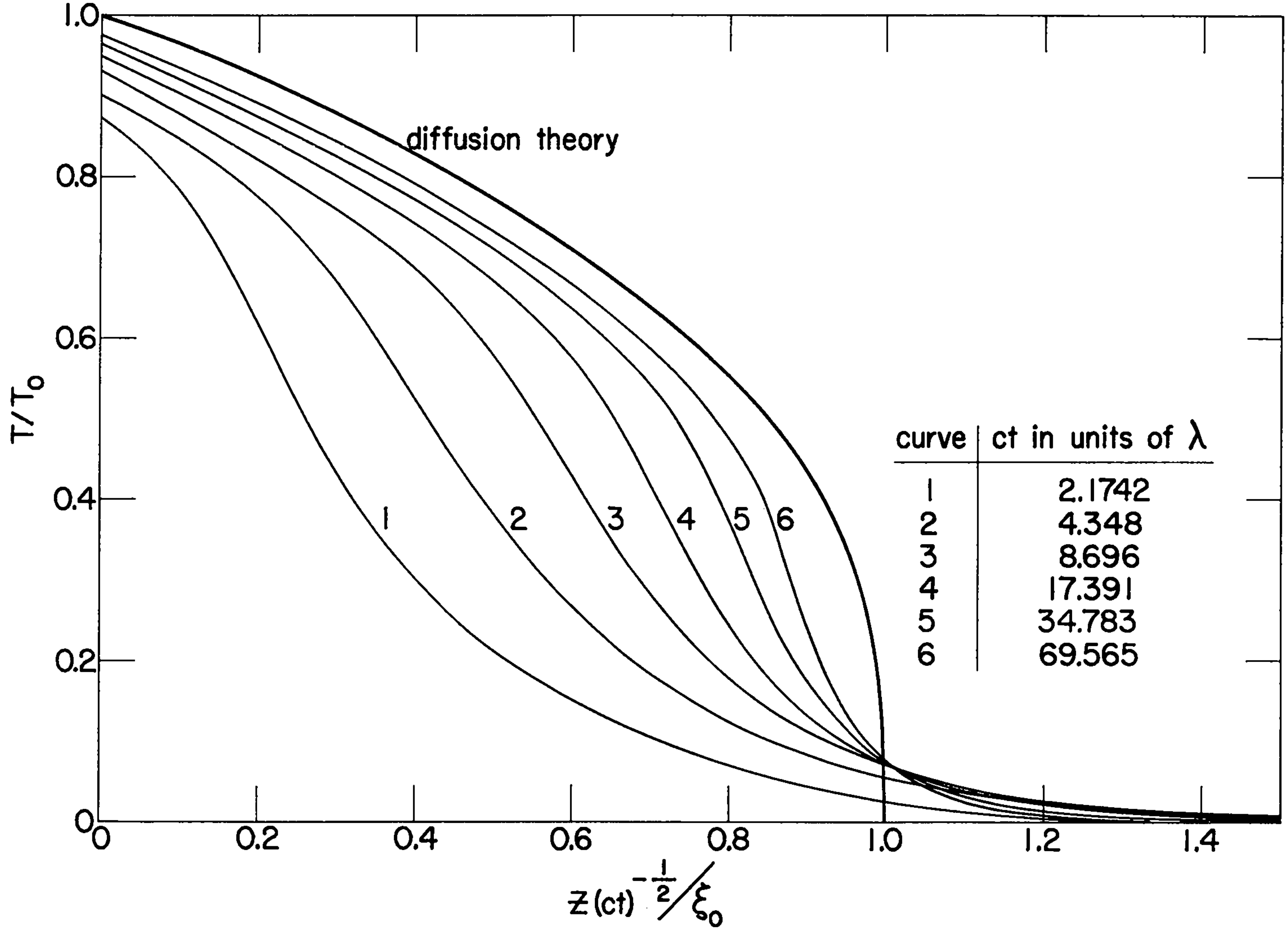

Fig. 5. Temperature wave form in diffusion and transport theory (constant mean free path case). ($aT_o{}^3/b = 1.690$). Unit for z and ct is λ. $\xi_o = z_o(ct)^{-1/2} = 1.060$.

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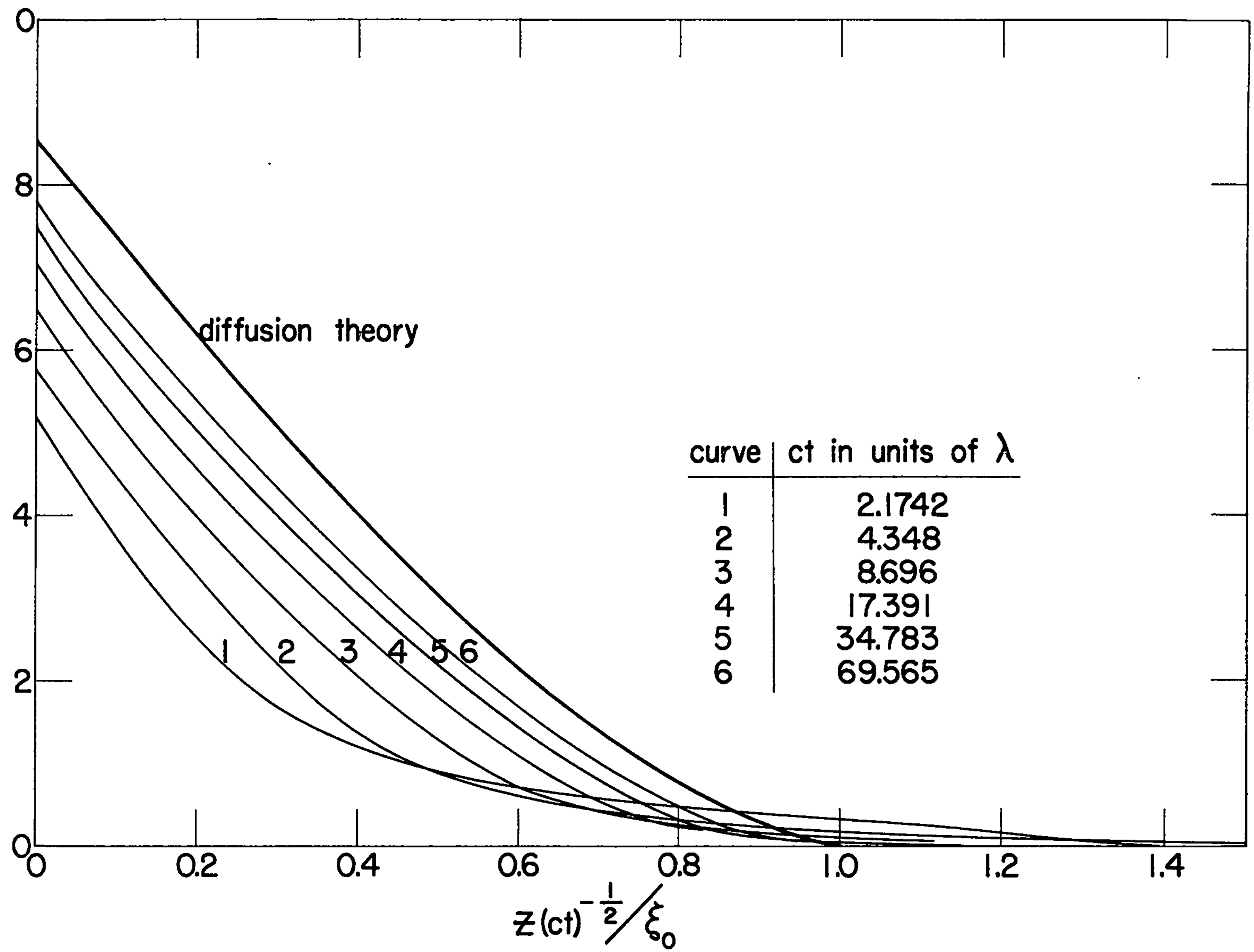

Fig. 6. Radiation energy density E_r (constant mean free path case). ($aT_o^3/b = 1.690$). Unit for E_r is .1168 aT_o^4.

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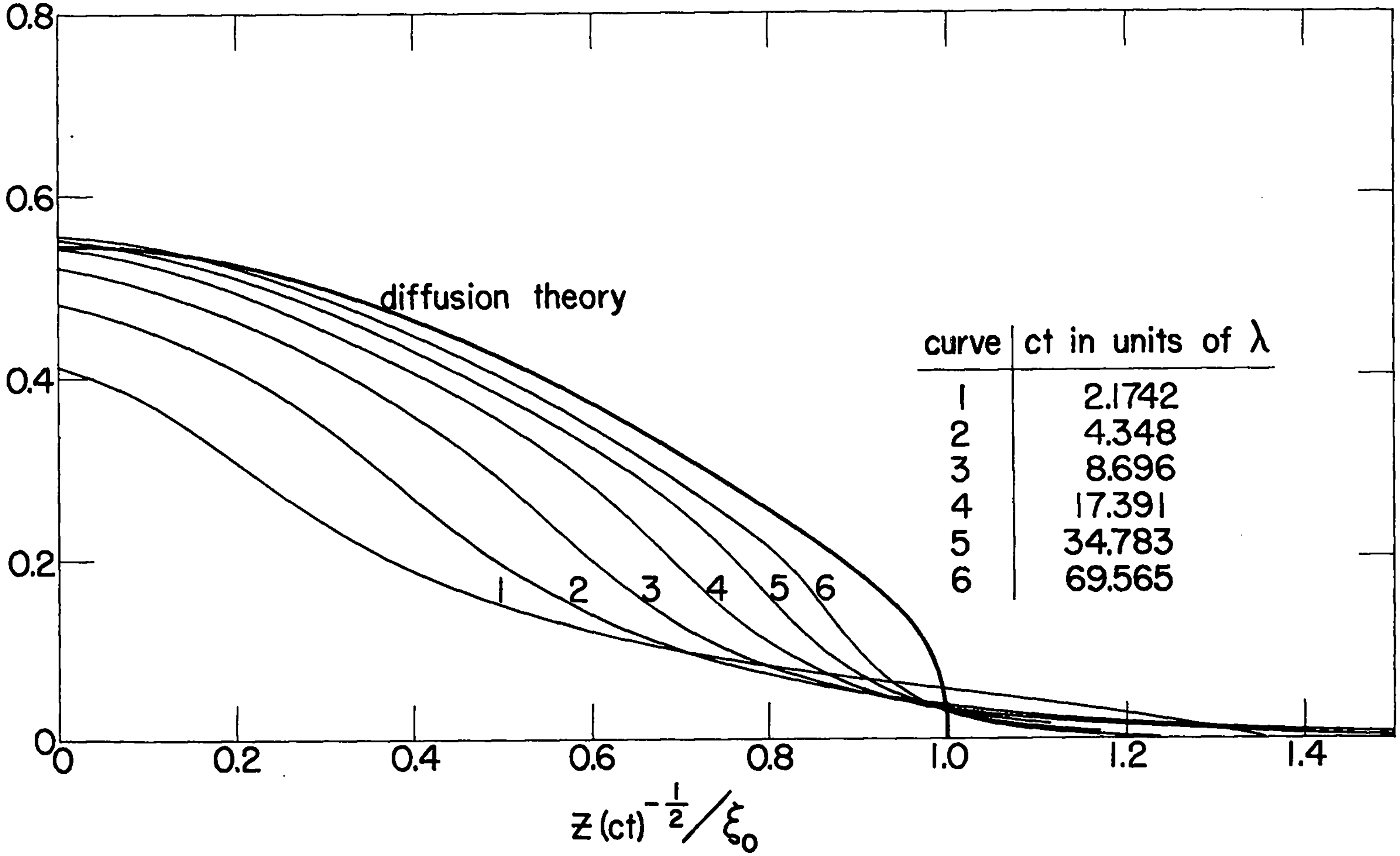

Fig. 7. Net flux (constant mean free path case). ($aT_o^3/b = 1.690$). The curves represent the quantity $(ct)^{1/2}F(z,t)$ measured in units $.3247\,\lambda^{1/2}caT_o^4/4$.

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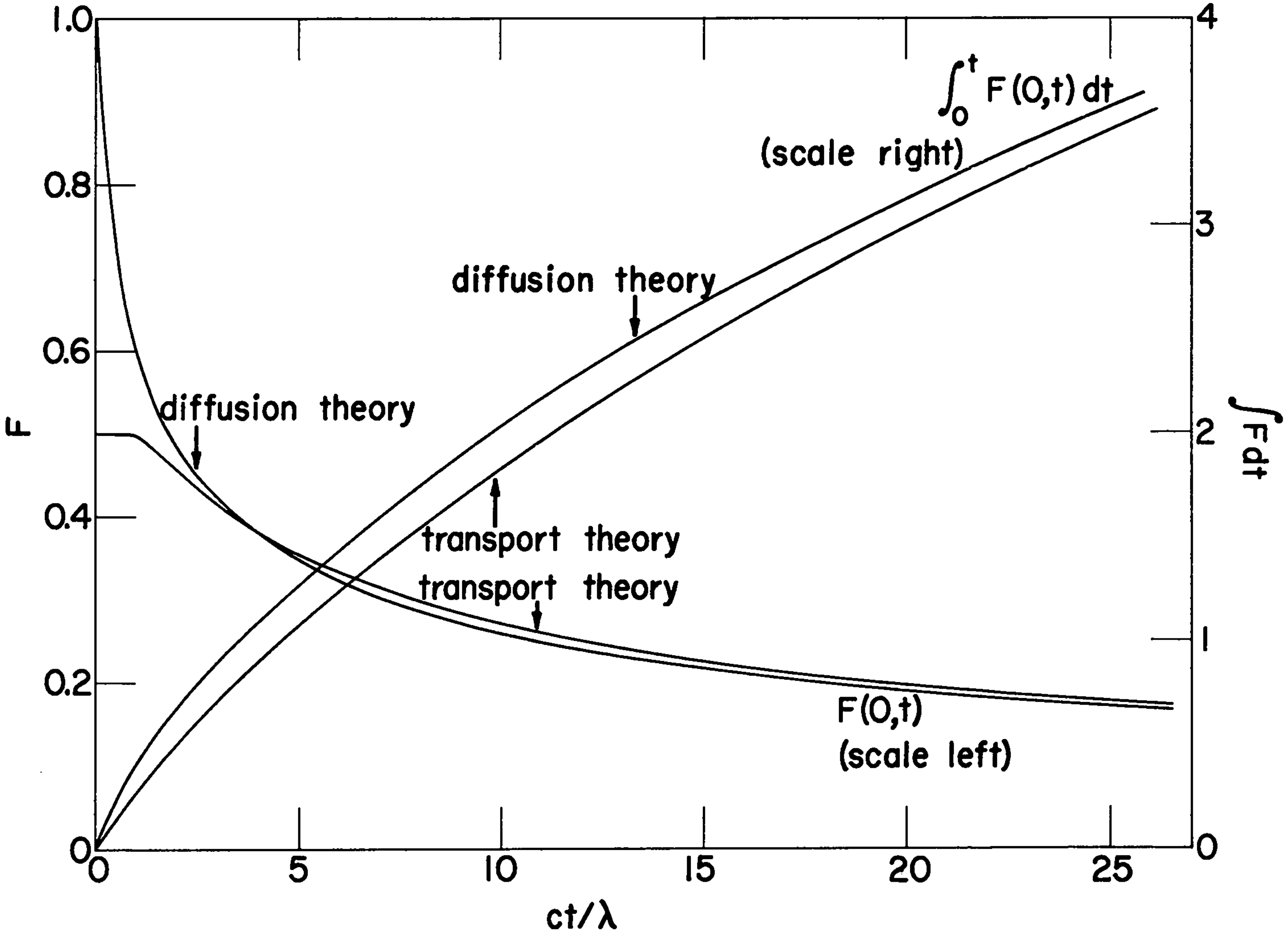

Fig. 8. Energy penetration and net flux through boundary (constant mean free path case). ($aT_o^3/b = 1.690$). Forward current across boundary prescribed for diffusion case. Unit for $\int F dt$ is $aT_o^4\lambda$; unit for F is $caT_o^4/2$. (Compare Fig. 1, Fig. 4.)

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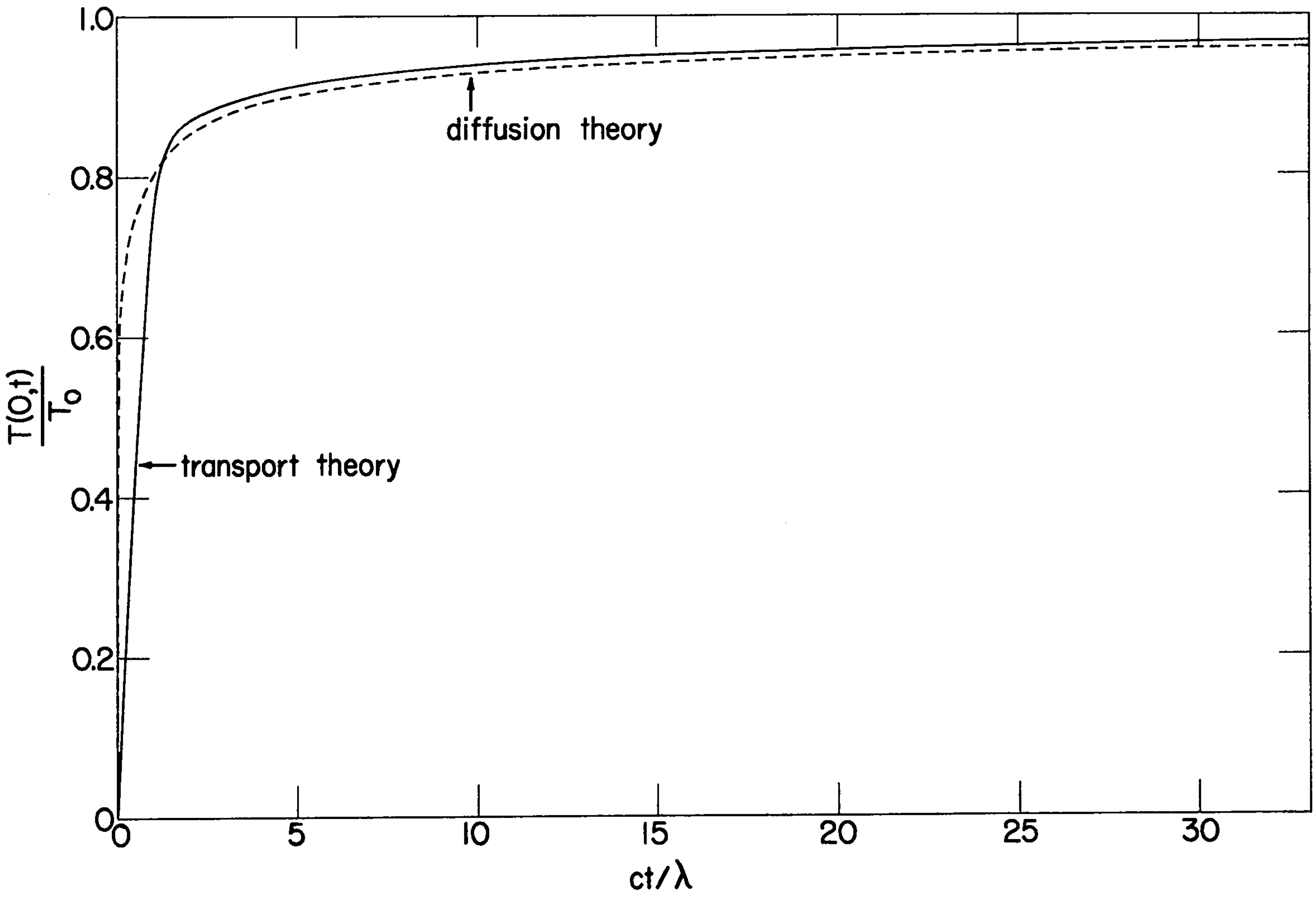

Fig. 9. Surface temperature of slab vs time (constant mean free path case). Forward current across boundary prescribed for diffusion theory. ($aT_o^3/b = 1.690$).

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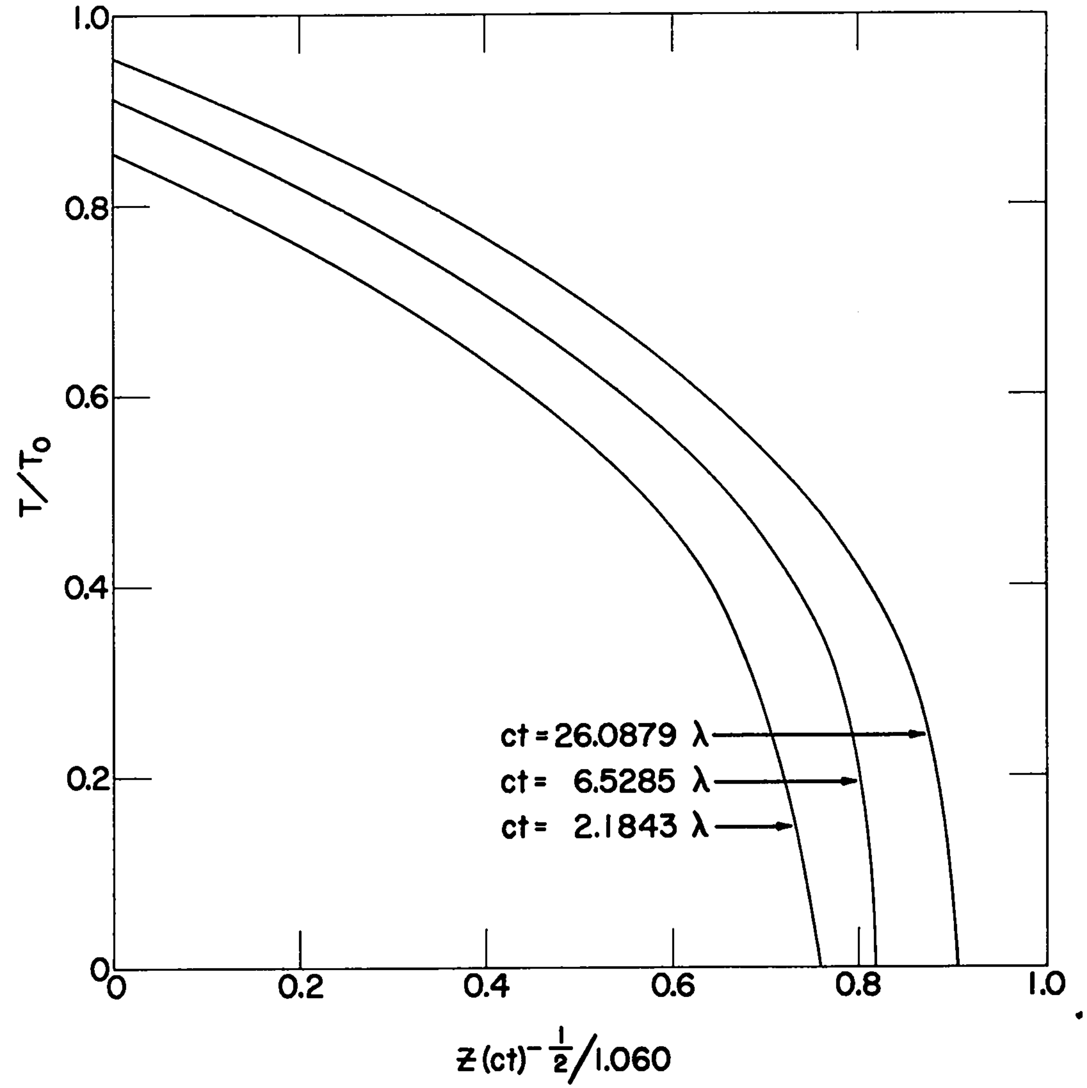

Fig. 10. Shape of diffusion theory temperature wave at selected times (constant mean free path case; forward current prescribed at boundary). ($aT_o^3/b = 1.690$). Compare Fig. 5.

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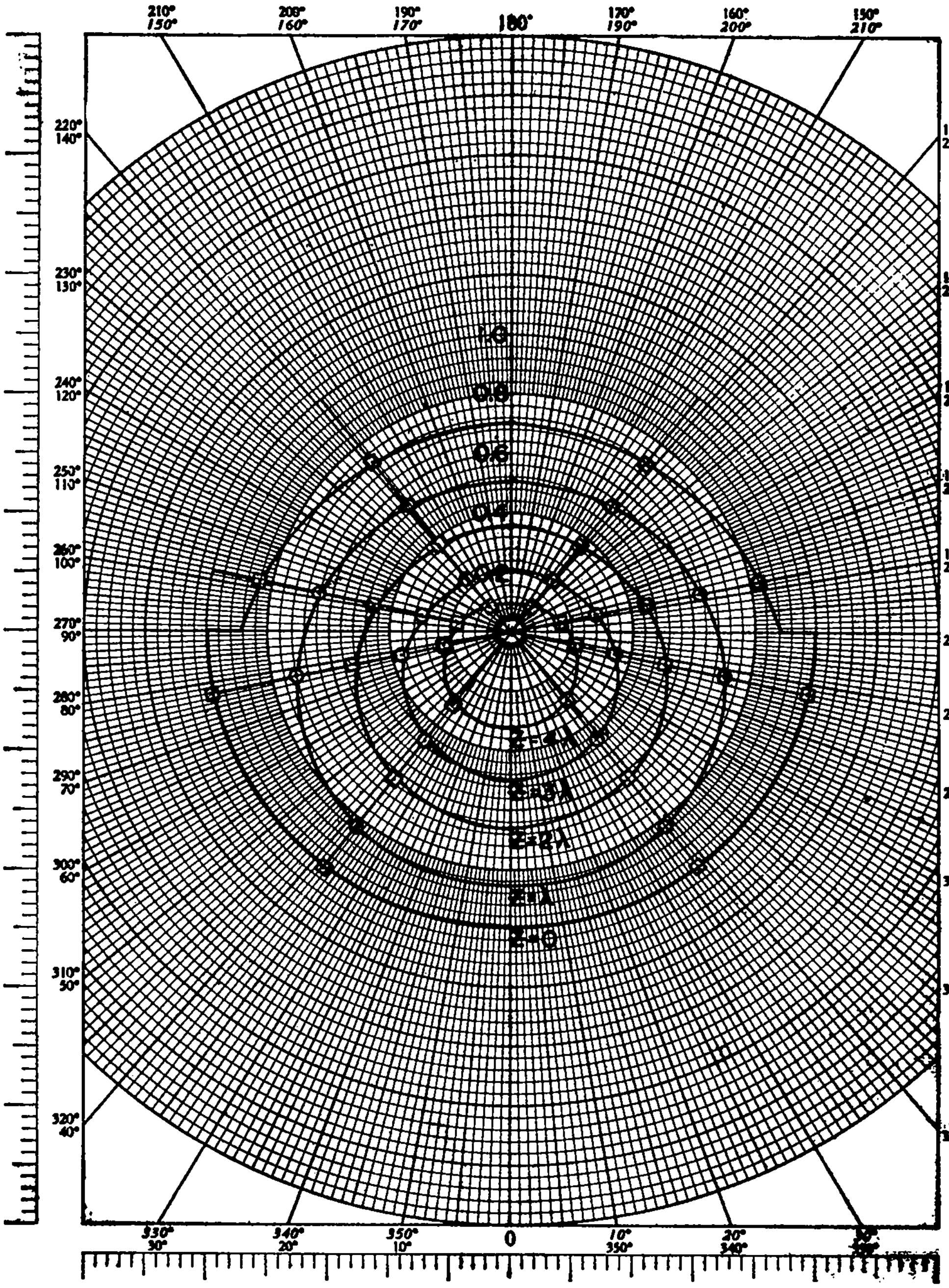

Fig. 11. Example of angular distribution at various depths as calculated by transport theory for constant mean free path case. ($aT_o^3/b = 1.690$; $ct/\lambda = 43.5$).

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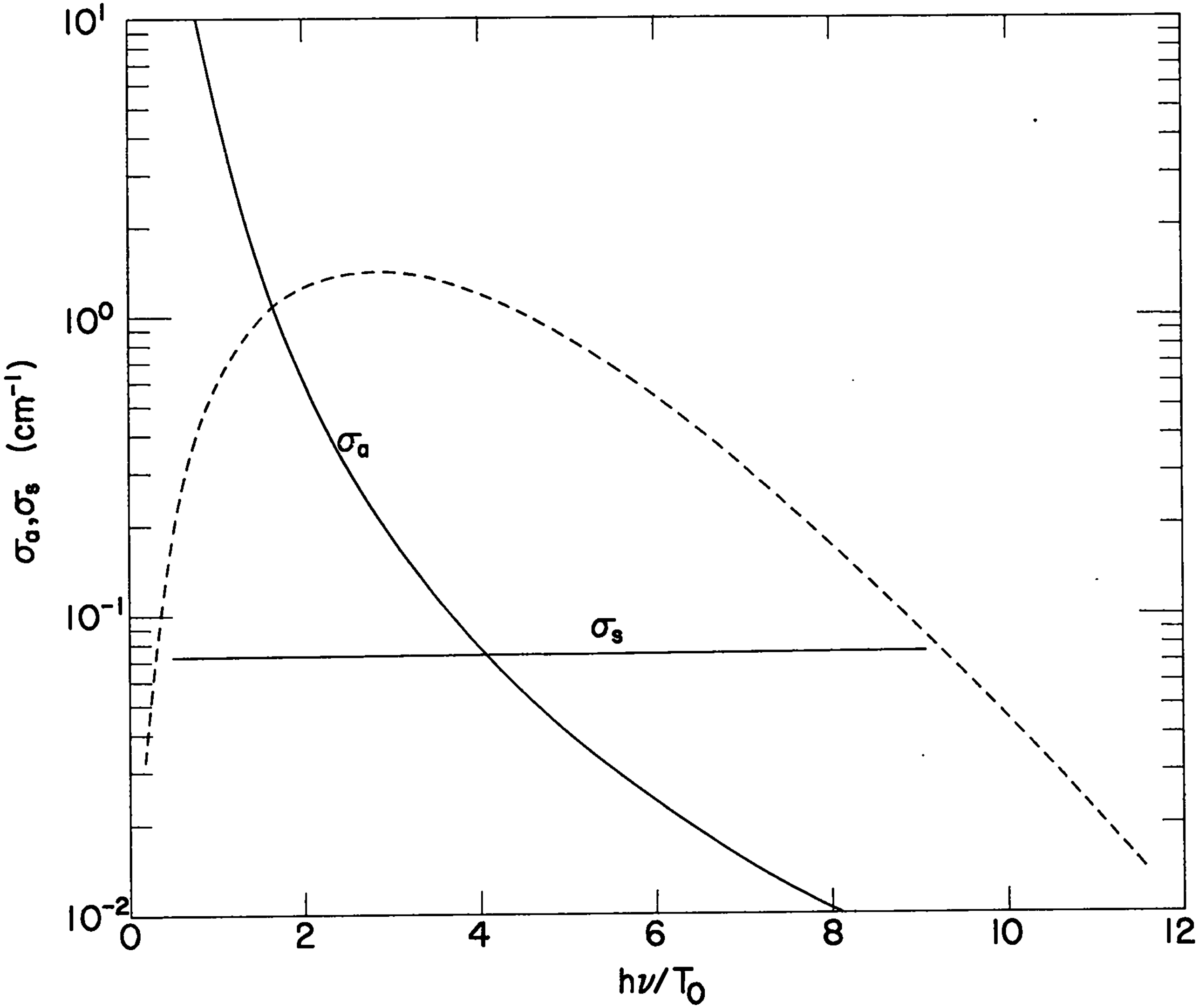

Fig. 12. Absorption and scattering coefficients for boron. (Also shown is Planck distribution corresponding to temperature T_o = .7.)

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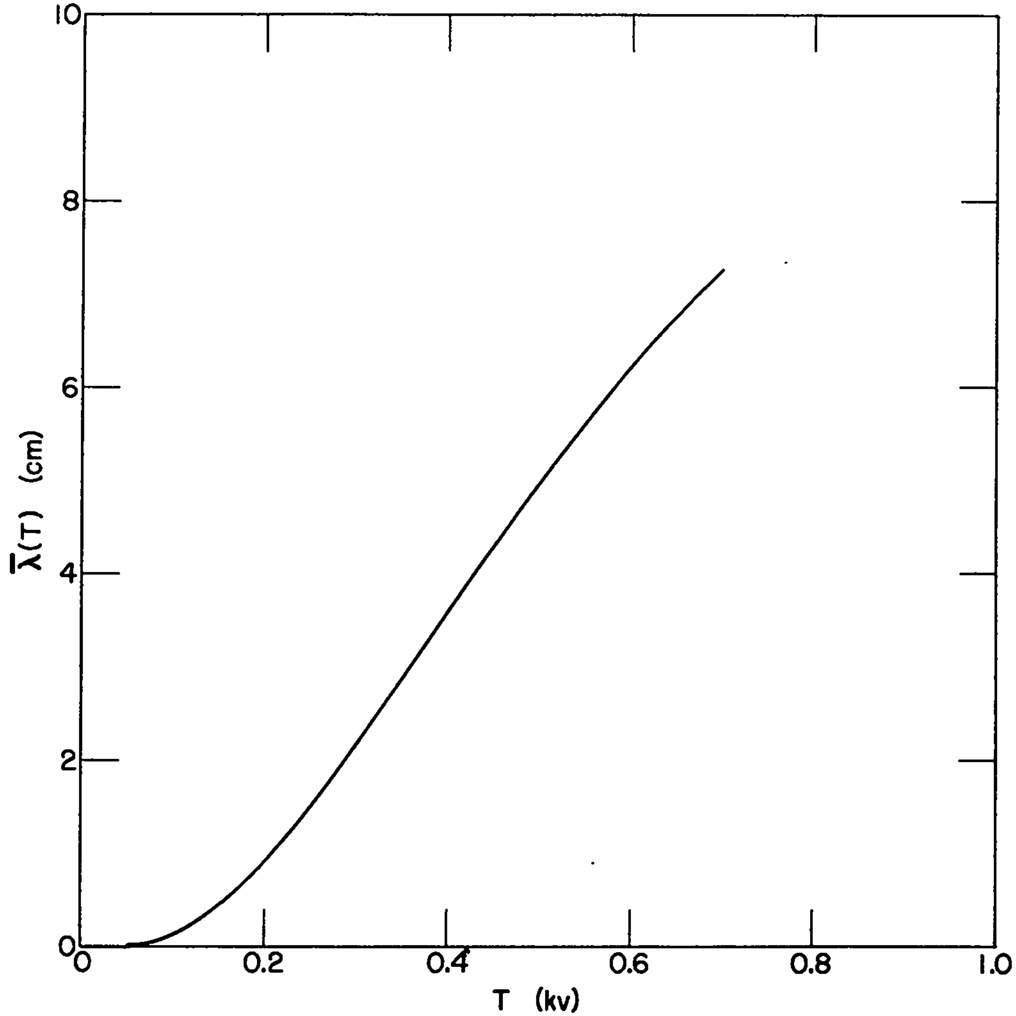

Fig. 13. Rosseland mean $\bar{\lambda}(T)$ per boron.

$$\bar{\lambda}(T) = \begin{cases} .15799 - 5.5478T + 60.796T^2 - 76.0448T^3 + 30.293T^4 & .05 \leq T \leq .7 \\ 0 & 0 \leq T < .05 \end{cases}$$

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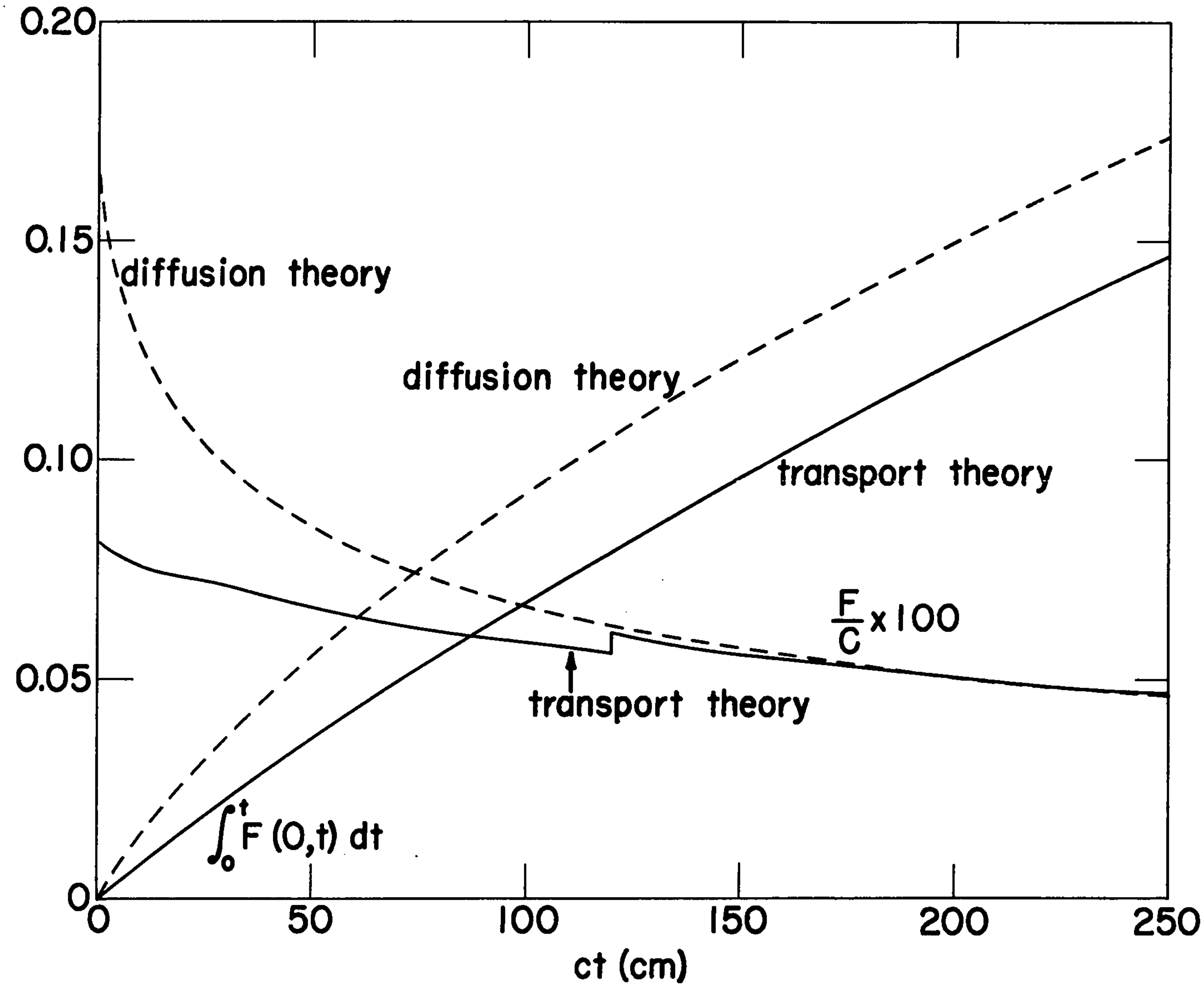

Fig. 14. Energy penetration and net flux through boundary of boron slab (c.g.s. units x 10^{16}).

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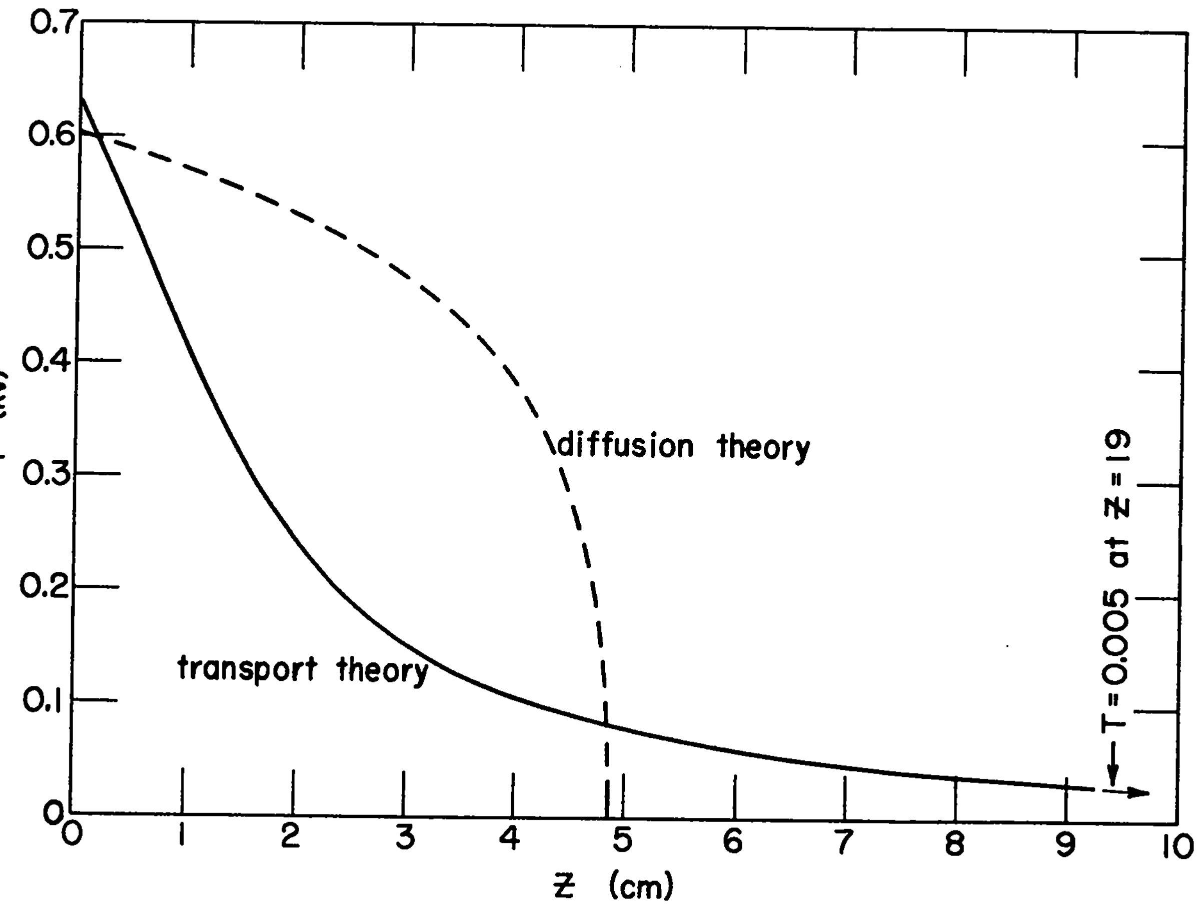

Fig. 15. Profile of temperature wave in boron slab. ct = 75 cm. (E_m = .028426 x 10^{16}T).

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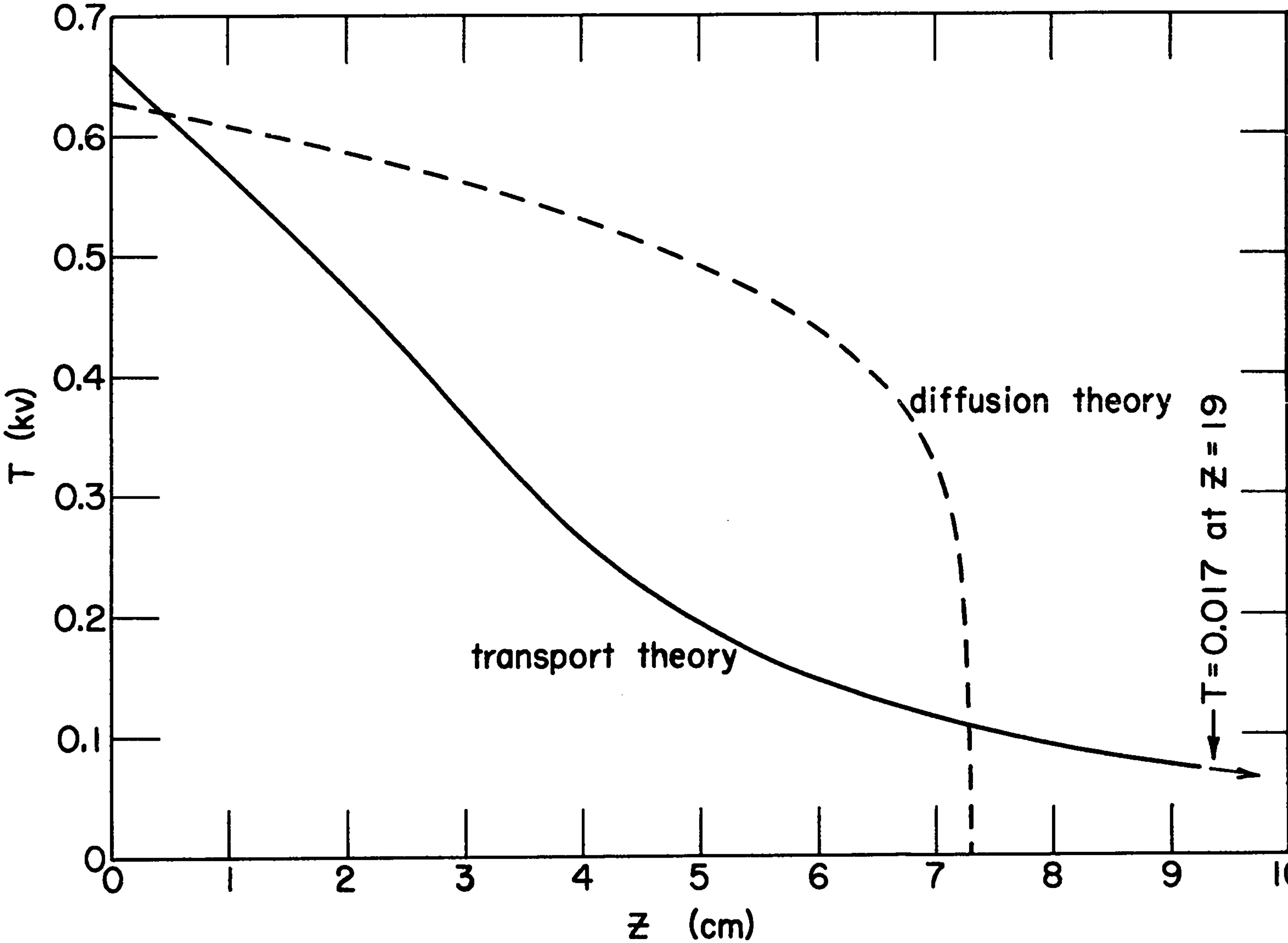

Fig. 16. Profile of temperature wave in boron slab. ct = 150 cm.

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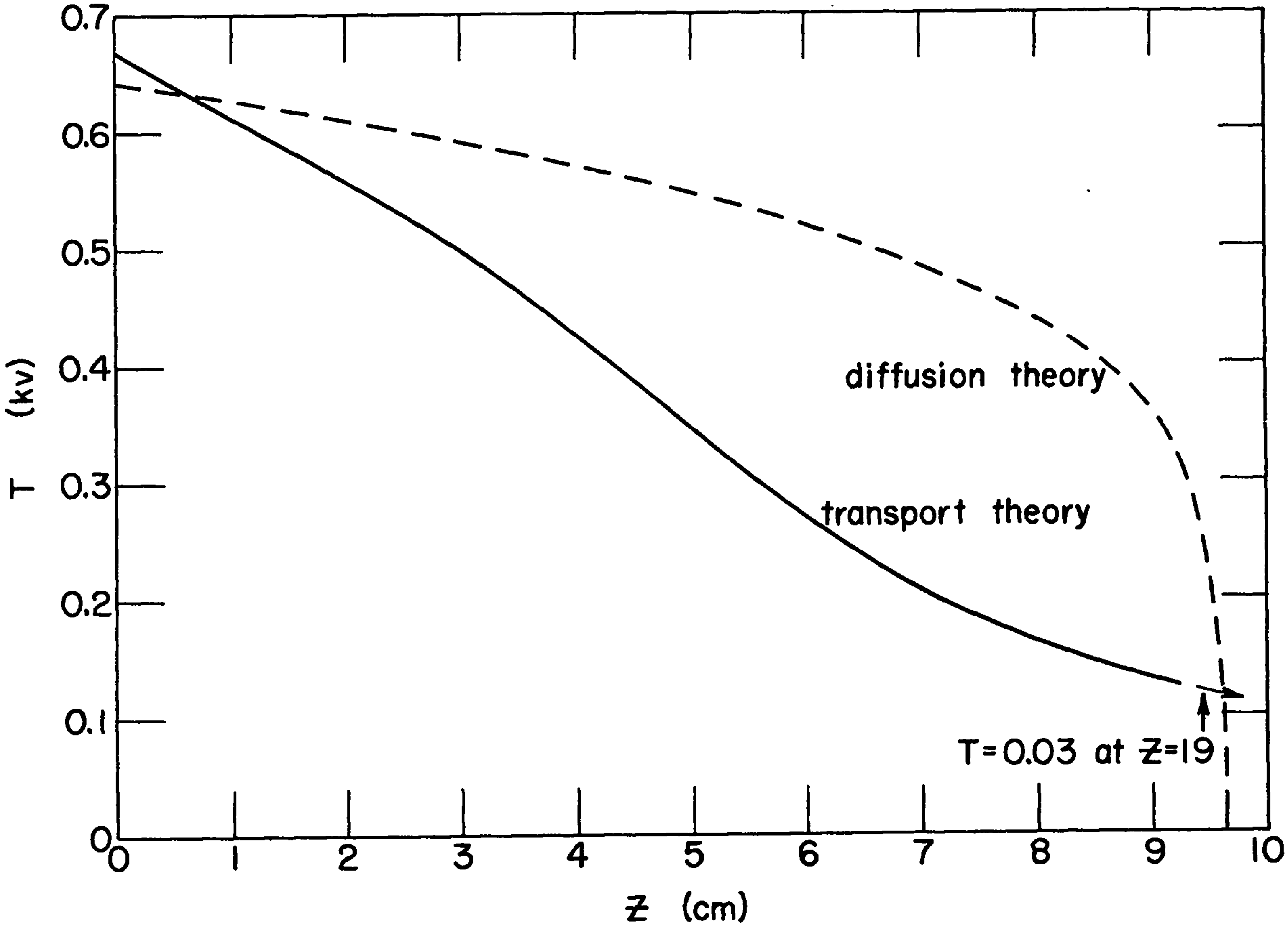

Fig. 17. Profile of temperature wave in boron slab. ct = 225 cm.

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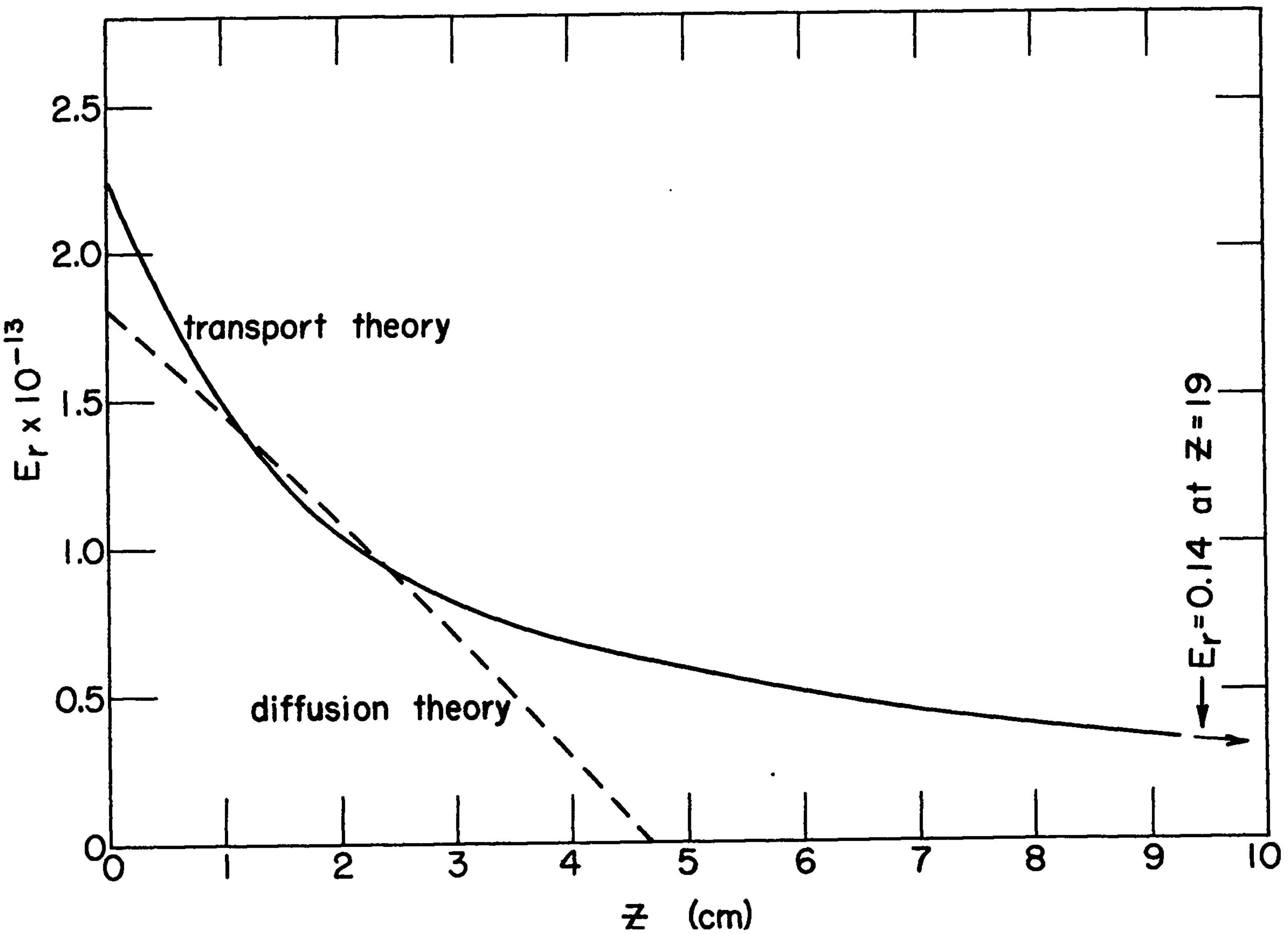

Fig. 18. Profile of radiation energy density wave in boron. ct = 75 cm.

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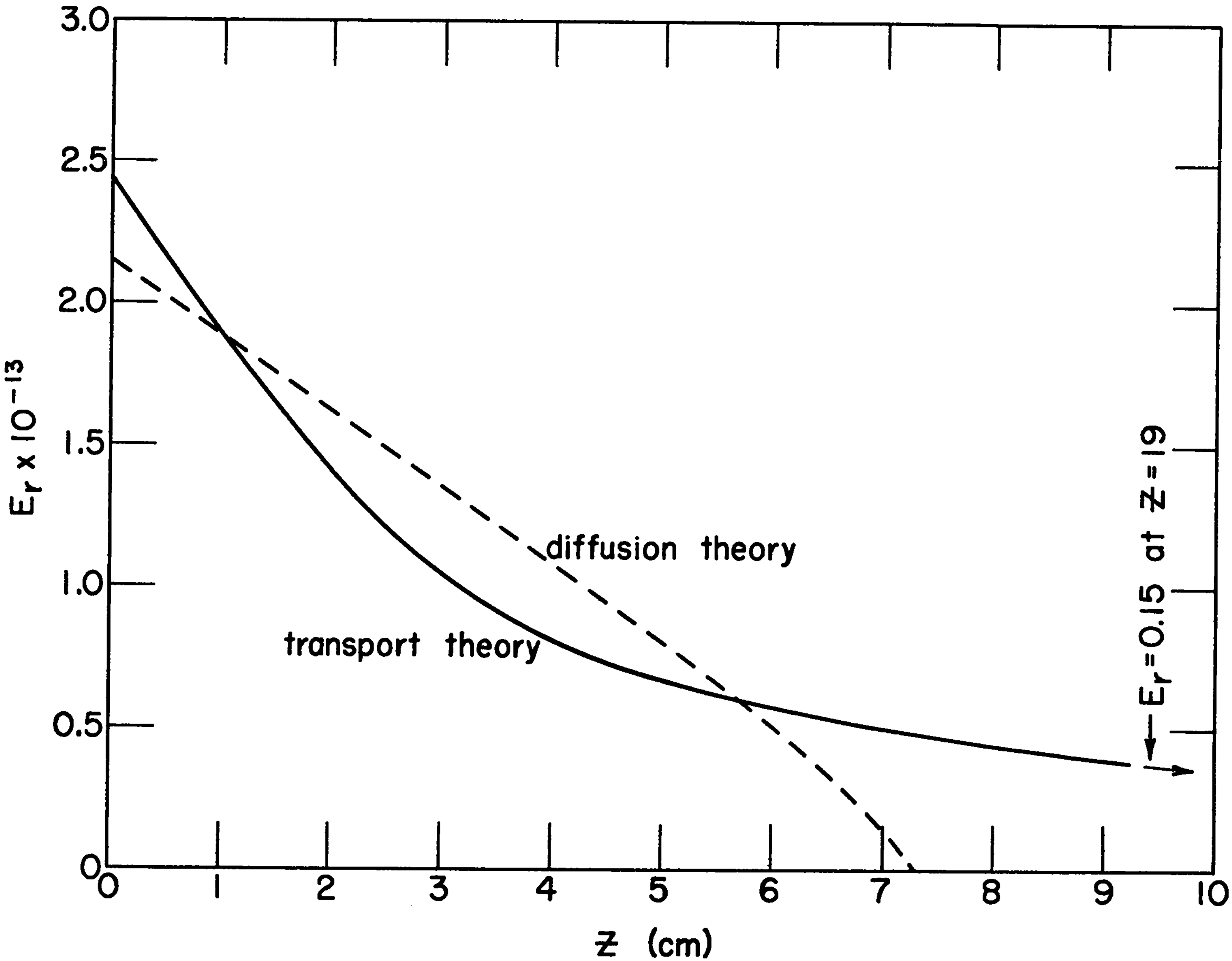

Fig. 19. Profile of radiation energy density wave in boron. ct = 150 cm.

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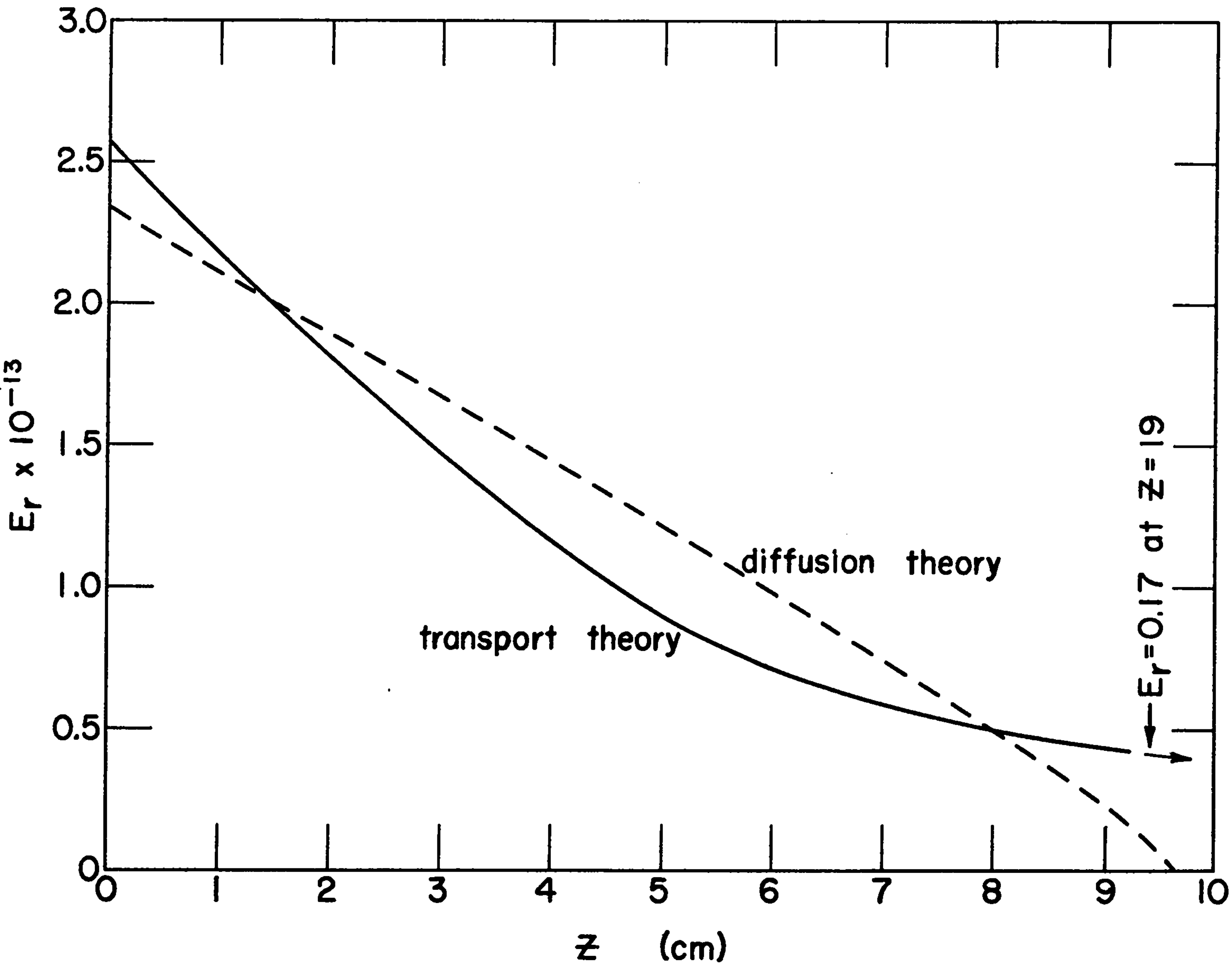

Fig. 20. Profile of radiation energy density wave in boron. ct = 225 cm.

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